Phys 410
Spring 2013
Lecture #15 Summary
22 October, 2013

We discussed several examples of constrained systems by the Lagrangian method. The key step is to identify the number of degrees of freedom in the problem and find the most efficient set of generalized coordinates. Examining the constraints in the system is often a good way to identify the appropriate generalized coordinates. Writing down the kinetic and potential energies in terms of these generalized coordinates is often facilitated by using Cartesian or cylindrical or spherical coordinates, and then converting completely to the generalized coordinates.

The first example was the problem of a frictionless block sliding down the side of a wedge of angle α which is sliding horizontally over a frictionless surface. Because the block and wedge are constrained to remain in contact, and the wedge and horizontal surface are also constrained to remain in contact, there are really only two degrees of freedom in this problem: the displacement of the wedge in the horizontal direction (q_2), and the displacement of the block down the wedge (q_1). The kinetic and potential energies can be written in terms of these coordinates and their time derivatives. We found that the horizontal component of momentum is conserved, and that the block moves down the wedge with a constant acceleration that depends of the mass of the block and wedge, as well as the angle α. The time for the block to reach the bottom of the wedge is just that of a particle moving with constant acceleration.

The rotating bead on a loop problem was then analyzed. A bead of mass m is constrained to move on a vertical circular loop of radius R, and the loop is set into rotation about the vertical axis through the loop center, at angular frequency ω. There is a single generalized coordinate θ, which is the angle that the bead makes with respect to the vertically-down direction from the center of the loop. There are two components of velocity for the bead, one around the loop ($v_\theta = R\dot{\theta}$) and the other around the vertical axis ($v_\varphi = \rho\omega = R\sin\theta\,\omega$). The Lagrangian is $\mathcal{L}(\theta,\dot{\theta}) = T - U = \frac{mR^2}{2}\left(\dot{\theta}^2 + \omega^2\sin\theta^2\right) - mgR(1-\cos\theta)$. The resulting equation of motion is $\ddot{\theta} = (\omega^2\cos\theta - g/R)\sin\theta$. This cannot be solved in closed form for $\theta(t)$. Note that the equation reduces to the equation of motion of a pendulum in the limit $\omega \to 0$.

Even though we cannot solve this equation for $\theta(t)$, we can learn much about the possible equilibrium solutions to the equation. From the in-class demonstration we showed that there are several different equilibrium points for the bead while the loop is rotating. The equilibrium points are those special angles θ_0 where a particle can be placed with no initial velocity $\dot{\theta} = 0$ and will stay there because the acceleration is zero, $\ddot{\theta} = 0$. The zeroes of the above equation of motion come from the two terms in the product on the RHS. The first are those for which $\sin\theta_0 = 0$, which include $\theta_0 = 0, \pi$. The position $\theta_0 = \pi$ is always unstable, while that for $\theta_0 = 0$ is stable for low angular velocities ω. The other equilibrium points are given by the zero of the term in parentheses: $\cos\theta_0 = g/\omega^2 R$. However, since the magnitude of $\cos\theta_0$ is bounded, this requires a certain minimum angular velocity, or greater, to be satisfied: $\omega \geq \sqrt{g/R}$. There are two equilibrium angles in this case: $\theta_0 = \pm\cos^{-1}(g/\omega^2 R)$, both of which are stable when they exist. In summary, the angle $\theta_0 = 0$ is stable for $\omega < \sqrt{g/R}$, and it bifurcates (becomes unstable) into two other stable points at $\theta_0 = \pm\cos^{-1} g/\omega^2 R$. In the limit as $\omega \to \infty$, the angles become $\theta_0 = \pm\pi/2$, which is the 'outside' of the circular hoop.

If a generalized coordinate does not appear in the Lagrangian it is said to be ignorable or cyclic. The corresponding generalized momentum is conserved.

Finally we derived a new quantity known as the Hamiltonian. The Lagrangian was engineered specifically to reproduce Newton's second law in component form, however it does not have a simple physical interpretation. By taking the total time derivative of the Lagrangian we could create a new quantity $\mathcal{H}$ that is time-invariant, subject to the condition that $\frac{\partial\mathcal{L}}{\partial t} = 0$, and it is found to be $\mathcal{H} = \sum_{i=1}^{n} p_i\,\dot{q}_i - \mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. If, in addition, there is a time-independent relationship between the Cartesian coordinates and the generalized coordinates, $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \ldots q_i, \ldots q_n)$, then the Hamiltonian has a simple interpretation as the total mechanical energy $T + U$.